JOURNAL OF ALGEBRA 18, 377–383 (1971)

On Free Products with Amalgamation of Two Infinite Cyclic Groups

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Communicated by Marshall Hall, Jr.

Received February 23, 1970

Let F_n be the free group with free generating set $x_1 ,..., x_n$ and let G be a group with one defining relation given by the presentation $\langle x_1 ,..., x_n ; R = 1\rangle$. We denote by $P(G)$ the set of all presentations of G of the form $\langle x_1 ,..., x_n ; S = 1\rangle$. Two elements $\langle x_1 ,..., x_n ; S = 1\rangle$ and $\langle x_1 ,..., x_n ; Q = 1\rangle$ of $P(G)$ are said to be N-equivalent if there exists an automorphism ϕ of F_n such that $\phi S = Q^{\pm 1}$. This defines an equivalence relation on $P(G)$; the number of equivalence classes under this relation is denoted by $| P(G)|$. It has been conjectured by W. Magnus [2, p. 401] that $| P(G)| = 1$ for every group G with one defining relation. We show in Theorem 1 that this conjecture is false.

The counterexamples we obtain to the Magnus conjecture are certain $G_{k,l}$ groups, where, if k and l are integers different from 0 and ± 1, $G_{k,l}$ is the group defined by the presentation $\langle x_1 , x_2 ; x_1{}^k = x_2{}^l\rangle$. Thus each $G_{k,l}$ is a nontrivial free product with amalgamation of two infinite cyclic groups. In Theorems 2 and 3 we obtain results which may be useful in determining the number of nonequivalent presentations of such a group.

We note, first, that the method of the proof of Proposition 6.3 of Ref. [3] can be used to give the following result:

LEMMA 1. *Let k, l, k_1 and l_1 be integers different from* 0, ± 1. *Then $G_{k,l}$ is isomorphic to G_{k_1, l_1} if and only if either $| k | = | k_1 |$ and $| l | = | l_1 |$, or $| k | = | l_1 |$ and $| l | = | k_1 |$.*

DEFINITION. Two elements w_1 and w_2 of F_n are said to be equivalent if and only if there exists an automorphism ϕ of F_n such that $\phi w_1 = w_2$. The element w of F_n is said to be of minimal length if the length $| w |$ of w is less than or equal to the length of any element equivalent to w.

LEMMA 2. *Let p, t, r, s and k be integers such that $p > 0$, $| t | \geqslant 2$,*

$|t+1| \geqslant 2$, $|r| \geqslant 2$, $|s| \geqslant 2$ *and* $|k| \geqslant 2$. *Then* $x_1{}^r x_2^{-s}$ *and* $(x_1{}^k x_2^{-t})^p x_2^{-1}$ *are both of minimal length in* F_n.

Proof. This follows immediately from Theorem 3 of Ref [4].

LEMMA 3. *Let* k *and* l *be integers different from* $0, \pm 1$ *and let* $l = pt + 1$ *for some integers* p *and* t. *Then* $G_{k,l}$ *has presentation* $\langle x_1, x_2; x_2 = (x_1{}^k x_2^{-t})^p \rangle$.

Proof.

$$\begin{aligned} G_{k,l} &\cong \langle x_1, y; x_1{}^k = y^{pt+1} \rangle, \\ &\cong \langle x_1, x_2, y; x_1^k y^{-pt} = y, x_2 = y^p \rangle, \\ &\cong \langle x_1, x_2, y; x_1{}^k x_2^{-t} = y, x_2 = y^p \rangle, \\ &\cong \langle x_1, x_2; x_2 = (x_1{}^k x_2^{-t})^p \rangle. \end{aligned}$$

DEFINITION. We denote by T the set of integers i such that either $|i| \leqslant 3$ or both $i-1$ and $i+1$ are primes.

THEOREM 1. *Let* k *and* l *be integers such that* $|k| \geqslant 2$, $|l| \geqslant 2$ *and at least one of* k, l *is not an element of* T. *Then* $|P(G_{k,l})| > 1$.

Proof. Since $P(G_{k,l}) = P(G_{l,k})$, we can assume that $l \notin T$. We suppose, first, that $l-1$ is not a prime. Let p be the least positive prime dividing $l-1$, and let t be such that $l-1 = pt$. Then $|t| \geqslant 2$, and also $|t+1| \geqslant 2$, since otherwise we would have $-2 \leqslant t \leqslant 0$, so that $t = -2$ and $p = 2$; but then $l = -3$, which is a contradiction since $l \notin T$. It follows from Lemma 2 that $(x_1{}^k x_2^{-t})^p x_2^{-1}$ is of minimal length in F_2. Now $|x_1{}^k x_2^{-l}| = |k| + |l|$, and $|(x_1{}^k x_2^{-t})^p x_2^{-1}| = |k|p + |l|$, since if $t > 0$ then

$$|(x_1{}^k x_2^{-t})^p x_2^{-1}| = |k|p + tp + 1 = |k|p + |l|,$$

while if $t < 0$, then

$$|(x_1{}^k x_2^{-t})^p x_2^{-1}| = |k|p - tp - 1 = |k|p + |l|.$$

It follows that $(x_1{}^k x_2^{-t})^p x_2^{-1}$ is not equivalent to $(x_1{}^k x_2^{-l})^{\pm 1}$ in F_2, since these elements are minimal and have different lengths. Hence the elements $\langle x_1, x_2; x_1{}^k x_2^{-l} = 1 \rangle$ and $\langle x_1, x_2; (x_1{}^k x_2^{-t})^p x_2^{-1} = 1 \rangle$ of $P(G_{k,l})$ are not N-equivalent.

Now suppose that $l-1$ is a prime. Then since $l \notin T$ we must have $-l \notin T$ and $-l-1$ is not a prime. From the special case proved above it follows that $|P(G_{k,-l})| > 1$, and so $|P(G_{k,l})| > 1$, since $P(G_{k,l}) = P(G_{k,-l})$. This proves the theorem.

Corollary 1. *Let k and l satisfy the conditions of the theorem. Then there is a presentation $\langle x_1, x_2 ; R = 1\rangle$ of $G_{k,l}$ such that no automorphic image of R in F_2 is of the form $x_1{}^r x_2^{-s}$.*

Proof. As in the proof of the theorem, it is sufficient to consider the case $l \notin T$ and $l - 1$ is not a prime. Let p and t be chosen as before, so that $G_{k,l}$ has presentation $\langle x_1, x_2 ; (x_1{}^k x_2^{-t})^p x_2^{-1} = 1\rangle$. Now if $(x_1{}^k x_2^{-t})^p x_2^{-1}$ is equivalent to $x_1{}^r x_2^{-s}$, then $G_{k,l}$ is isomorphic to $G_{r,s}$, and so, by Lemma 1, either $|r| = |k|$ and $|s| = |l|$, or $|r| = |l|$ and $|s| = |k|$. It follows easily from this that $x_1{}^r x_2^{-s}$ is equivalent to $x_1{}^k x_2^{-l}$, and so $(x_1{}^k x_2^{-t})^p x_2^{-1}$ is equivalent to $x_1{}^k x_2^{-l}$. This is a contradiction.

It has been shown by Shenitzer [3] that if G has presentation $\langle x_1, ..., x_n ; R = 1\rangle$ and G is a nontrivial free product, then there is an automorphism ϕ of F_n such that ϕR does not involve all of the free generators $x_1, ..., x_n$. Now Corollary 1 gives examples of groups $\langle x_1, ..., x_n ; R = 1\rangle$ which are nontrivial free products of two free groups with an infinite cyclic subgroup amalgamated, even though no automorphic image of R in F_n has the form $w_1 w_2$, where w_1 and w_2 are nonidentity elements of F_n which have no free generator x_i in common; thus the obvious generalization of Shenitzer's result to free products with amalgamation is false.

Corollary 2. *Let m be any positive integer. Then we can choose k and l so that $|P(G_{k,l})| \geqslant m$.*

Proof. Choose $k \geqslant 2$ and $l = p_1 p_2 \cdots p_m + 1$, where $p_1, ..., p_m$ are the first m positive primes. Let t_i be such that $l = p_i t_i + 1$ $(i = 1, 2, ..., m)$. Then $R_i = (x_1{}^k x_2^{-t_i})^{p_i} x_2^{-1}$ is of minimal length in F_2, and its length is $kp_i + l$, so that R_i is not equivalent to $R_j^{\pm 1}$ if $i \neq j$, since $kp_i + l \neq kp_j + l$. Now $\langle x_1, x_2 ; R_i = 1\rangle$ is an element of $P(G_{k,l})$ $(i = 1, 2, ..., m)$, so that $|P(G_{k,l})| \geqslant m$.

We now show that Grusko's Theorem (see, e.g., [1]) can be extended to a class of groups which includes the $G_{k,l}$ groups.

Definition. Let G be a finitely generated group and let $H = \{h_1, ..., h_n\}$ and $K = \{k_1, ..., k_n\}$ be generating sets of G. The sets H and K are said to be Nielsen-equivalent if there exists an automorphism ϕ of F_n, with $\phi x_i = w_i(x_1, ..., x_n)$, say, such that $k_i = w_i(h_1, ..., h_n)$ $(i = 1, 2, ..., n)$.

We have:

Theorem 2. *Let G_1 and G_2 be finitely generated groups with normal subgroups L_1 and L_2 respectively, such that L_1 and L_2 are isomorphic. Let P be the free product of G_1 and G_2, amalgamating the subgroups L_1 and L_2. If $H = \{h_1, ..., h_n\}$ is a generating set of P, then H is Nielsen-equivalent to* *a generating set* $K = \{k_1, ..., k_n\}$ *of* P, *where each* k_i *has length either one or zero in* P.

Proof. Let P_1 be the free product of the groups G_1/L_1 and G_2/L_2, and let θ be the natural homomorphism from P to P_1. Then $\{\theta h_1, ..., \theta h_n\}$ is a generating set of P_1, and by Grusko's Theorem this set is Nielsen-equivalent to a generating set $\{a_1, ..., a_n\}$ of P_1, where each a_i has length either one or zero in P_1. Let ϕ be an automorphism of F_n, with $\phi x_i = w_i(x_1, ..., x_n)$, say, such that $a_i = w_i(\theta h_1, ..., \theta h_n)$ $(i = 1, 2, ..., n)$. Then the set $\{k_1, ..., k_n\}$, where $k_i = w_i(h_1, ..., h_n)$ $(i = 1, 2, ..., n)$, is a generating set of P which is Nielson-equivalent to $\{h_1, ..., h_n\}$. We show that each k_i has length either one or zero in P.

Suppose that some k_i has length r greater than one. Let $T_1 \cup \{1\}$ and $T_2 \cup \{1\}$ be coset representative systems of L_1 in G_1 and L_2 in G_2, respectively. Then we can write $k_i = kb_1 \cdots b_r$, where $k \in L_1$, $b_j \in T_1 \cup T_2$ $(j = 1, 2, ..., r)$, and $b_j \in T_1$ if and only if $b_{j+1} \in T_2 (j = 1, 2, ..., r - 1)$. Thus

$$\theta k_i = \theta k \theta b_1 \cdots \theta b_r = \theta b_1 \cdots \theta b_r,$$

since $k \in \ker \theta$. Now $\theta b_j \neq 1$, since $b_j \notin \ker \theta$ $(j = 1, 2, ..., r)$. Hence θk_i has length r in P_1, since each θb_j has length one, and $\theta b_j \in G_1/L_1$ if and only if $\theta b_{j+1} \in G_2/L_2$ $(j = 1, 2, ..., r - 1)$. This is a contradiction, since $\theta k_i = a_i$ has length either one or zero in P_1. This proves the theorem.

Remark. We note that we regard each presentation $\langle x_1, ..., x_n ; R_1 = 1, R_2 = 1, ... \rangle$ as defining a specific group G, the elements of G being the cosets of the normal closure $\{R_1, R_2, ...\}^{F_n}$ of the set $\{R_1, R_2, ...\}$ in F_n; by a presentation of G we mean a presentation of a group isomorphic to G (of course, if $\{S_1, S_2, ...\}$ is a subset of F_n such that $\{S_1, S_2, ...\}^{F_n} = \{R_1, R_2, ...\}^{F_n}$, the group defined by the presentation $\langle x_1, ..., x_n ; S_1 = 1, S_2 = 1, ... \rangle$ is actually G, and not just an isomorphic copy of G). We say that the presentation $\langle x_1, ..., x_m ; Q_1 = 1, Q_2 = 1, ... \rangle$ of G can be obtained from, or it corresponds to the generating set $\{h_1, ..., h_m\}$ of G, if $\ker \psi = \{Q_1, Q_2, ...\}^{F_m}$, where ψ is the homomorphism from F_m to G such that $\psi x_i = h_i$ $(i = 1, 2, ..., m)$. Clearly, every presentation of G can be obtained from some generating set of G, and two presentations $\langle x_1, ..., x_m ; S_1 = 1, S_2 = 1, ... \rangle$ and $\langle x_1, ... x_m ; Q_1 = 1, Q_2 = 1, ... \rangle$ of G correspond to the same generating set of G if and only if $\{S_1, S_2, ...\}^{F_m} = \{Q_1, Q_2, ...\}^{F_m}$. We say that a generating set $\{h_1, ..., h_n\}$ of G is a one relation generating set if there is a presentation $\langle x_1, ..., x_n ; R = 1 \rangle$ of G which corresponds to this set. Finally, we note that if $\{h_1, ..., h_n\}$ and $\{k_1, ..., k_n\}$ are Nielsen-equivalent generating sets of G, where $\phi : x_i \to w_i(x_1, ..., x_n)$ is an automorphism of F_n such that $k_i = w_i(h_1, ..., h_n)$

$(i = 1, 2,..., n)$, and $\langle x_1 ,..., x_n ; S_1 = 1, S_2 = 1,...\rangle$ is a presentation of G corresponding to the generating set $\{h_1 ,..., h_n\}$, then $\langle x_1 ,..., x_n ; \phi^{-1}S_1 = 1, \phi^{-1}S_2 = 1,...\rangle$ is a presentation of G corresponding to the generating set $\{k_1 ,..., k_n\}$.

Now $G_{k,l}$ is the group defined by the presentation $\langle x_1 , x_2 ; x_1{}^k x_2^{-l} = 1\rangle$; we denote by a_1 , a_2 the elements of $G_{k,l}$ corresponding to the generator symbols x_1 , x_2, respectively, in this presentation. From Theorem 2 and the above remarks it is easy to see that there is a transversal of the N-equivalence classes of $P(G_{k,l})$ such that each element of this transversal corresponds to a one relation generating set of $G_{k,l}$ of the form $\{a_1{}^r, a_2{}^s\}$. We now characterize those subsets of $G_{k,l}$ of the form $\{a_1{}^r, a_2{}^s\}$ which are generating sets of $G_{k,l}$.

THEOREM 3. *Let r and s be nonzero integers. Then $\{a_1{}^r, a_2{}^s\}$ is a generating set of $G_{k,l}$ if and only if $(r, s) = (r, k) = (s, l) = 1$.*

Proof. Suppose that $(r, s) = (r, k) = (s, l) = 1$. Then $(r, sk) = (s, rl) = 1$. Let $m_i , n_i (i = 1, 2)$ be integers such that $rm_1 + skn_1 = sm_2 + rln_2 = 1$. Then

$$a_1 = a_1^{rm_1} a_1^{skn_1} = (a_1{}^r)^{m_1}(a_2{}^s)^{ln_1},$$

since $a_1{}^k = a_2{}^l$, and similarly $a_2 = (a_1{}^r)^{kn_2}(a_2{}^s)^{m_2}$. Thus $\{a_1{}^r , a_2{}^s\}$ is a generating set of $G_{k,l}$.

Now suppose that $\{a_1{}^r , a_2{}^s\}$ is a generating set of $G_{k,l}$. Then for each integer m there is an expression for $a_1{}^m$ of the form

$$a_1^m = \prod_{i=1}^{p} a_1^{rr_i} a_2^{ss_i}. \tag{A}$$

Using the fact that $a_1{}^k (= a_2{}^l)$ is in the center of $G_{k,l}$, we can obtain from Eq. (A) an expression of the form

$$a_1^m = a_1^{rt_1} a_2^{st_2} \prod_{i=1}^{q} a_1^{rr_i'} a_2^{ss_i'}, \tag{B}$$

where $q \geqslant 0$, k divides rt_1, l divides st_2, no r_i' or s_i' is zero, except possibly r_1' or s_q', k does not divide rr_i' (if $r_i' \neq 0$) and l does not divide ss_i' (if $s_i' \neq 0$) $(i = 1, 2,..., q)$.

Let A_1 and A_2 be the subgroups of $G_{k,l}$ generated by a_1 and a_2, respectively. Then $G_{k,l}$ is the free product of these two subgroups, with the subgroup generated by $a_1{}^k$ amalgamated with the subgroup generated by $a_2{}^l$, the amalgamation being $a_1{}^k = a_2{}^l$. Hence in Eq (B), $a_1^{rt_1} a_2^{st_2}$ belongs to the amalgamated subgroup, while no nonidentity element of the form $a_1^{rr_i'}$ or $a_2^{ss_i'}$ belongs to this subgroup. Thus, taking $m = 1$ in Eq. (B), we see that we must have $q = 1$ and $s_1' = 0$ in this case. Let n be an integer such that $st_2 = nl$. Then we have

$$a_1 = a_1^{rt_1} a_2^{st_2} a_1^{rr_1'} = a_1^{r(t_1+r_1')} a_2^{nl} = a_1^{r(t_1+r_1')+kn}.$$

Since a_1 has infinite order in $G_{k,l}$, it follows that $r(t_1 + r_1') + kn = 1$, and so $(r, k) = 1$. Similarly, we can show that $(s, l) = 1$.

We now take $m = k$ in Eq. (B). Then we must have $q = 0$, so that $a_1^k = a_1^{rt_1} a_2^{st_2}$. Since $(r, k) = (s, l) = 1$, it follows that k divides t_1, and l divides t_2. Let p_1, q_1 be integers such that $t_1 = kp_1$ and $t_2 = lp_2$. Then

$$a_1^k = a_1^{rkp_1} a_2^{slp_2} = a_1^{krp_1+ksp_2}.$$

Hence $rp_1 + sp_2 = 1$, and so $(r, s) = 1$. This proves the theorem.

Note. Let $\{a_1^r, a_2^s\}$ be a generating set of $G_{k,l}$ and let m_i, n_i $(i = 1, 2)$ be chosen as in the theorem. Then the presentation

$$\langle x_1, x_2 ; (x_1^{m_1} x_2^{ln_1})^k (x_1^{kn_2} x_2^{m_2})^{-l} = 1, x_1^{-1}(x_1^{m_1} x_2^{ln_1})^r = 1, x_2^{-1}(x_1^{kn_2} x_2^{m_2})^s = 1\rangle \quad \text{(C)}$$

is a presentation of $G_{k,l}$ corresponding to the generating set $\{a_1^r, a_2^s\}$. To see this, we proceed as follows:

$$\begin{aligned}\langle x_1, x_2 ; x_1^k x_2^{-l} = 1\rangle &\cong \langle x_1, x_2, y_1, y_2 ; x_1^k x_2^{-l} = 1, y_1 = x_1^r, y_2 = x_2^s\rangle, \\ &\cong \langle x_1, x_2, y_1, y_2 ; x_1^k x_2^{-l} = 1, y_1 = x_1^r, y_2 = x_2^s, \\ &\qquad\qquad x_1 = y_1^{m_1} y_2^{ln_1}, x_2 = y_1^{kn_2} y_2^{m_2}\rangle, \\ &\cong \langle y_1, y_2 ; (y_1^{m_1} y_2^{ln_1})^k (y_1^{kn_2} y_2^{m_2})^{-l} = 1, \\ &\qquad\qquad y_1 = (y_1^{m_1} y_2^{ln_1})^r, y_2 = (y_1^{kn_2} y_2^{m_2})^s\rangle.\end{aligned}$$

Using presentation (C), it can be shown that the presentation

$$\langle x_1, x_2 ; x_1 x_2^l x_1^{-1} x_2^{-l} = 1, x_1^k x_2 x_1^{-k} x_2^{-1} = 1, x_1^{sk} x_2^{-rl} = 1\rangle$$

is also a presentation of $G_{k,l}$ corresponding to the generating set $\{a_1^r, a_2^s\}$. Thus $\{a_1^r, a_2^s\}$ is a one-relation generating set of $G_{k,l}$ if and only if there exists an element R of F_2 such that

$$\{R\}^{F_2} = \{x_1 x_2^l x_1^{-1} x_2^{-l}, x_1^k x_2 x_1^{-k} x_2^{-1}, x_1^{sk} x_2^{-rl}\}^{F_2}.$$

We have not been able to determine for which values of r and s this holds, apart from the special case when $\{a_1^r, a_2^s\}$ is Nielsen-equivalent to $\{a_1, a_2^p\}$, where $l = pt + 1$. In this case, $\{a_1^r, a_2^s\}$ is a one-relation generating set, since from Lemma 3 it follows that the presentation $\langle x_1, x_2; (x_1^k x_2^{-t})^p x_2^{-1}\rangle$ of $G_{k,l}$ corresponds to the generating set $\{a_1, a_2^p\}$.

We note finally that we have not been able to determine whether or not there exists a one-relator group G for which $|P(G)|$ is infinite; in particular, we do not know if this can be the case when G is a $G_{k,l}$ group.

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